1. Midterm 1

Due: In Lecture 10-21

Problem 1. Identify the set of real 2×2 matrices with $\mathbb{R}^4$, as in an earlier homework problem, and let M^3 denote the 3-dim'l submanifold of matrices of rank one. Find the tangent space to M^3 at the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Problem 2.
Define $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^1$ by

$$f(x, y, z) = z^2 x + e^z + y,$$

and note that $f(1, -1, 0) = 0$.

Use the Implicit Function Theorem to conclude that there is a function $g : \mathbb{R}^2 \to \mathbb{R}^1$ defined in some neighborhood of $(1, -1)$ such that $g(1, -1) = 0$ and such that

$$f(x, y, g(x, y)) = 0$$

for all (x, y) in that neighborhood.

Then calculate $\frac{\partial g}{\partial x}(1, -1)$ and $\frac{\partial g}{\partial y}(1, -1)$.

Problem 3. Prove that if S^k has non-vanishing vector field, then its antipodal map is homotopic to the identity.

Problem 4. Prove the following from basic principles (i.e. do not appeal to Sard's theorem)

a) Show that the natural copy of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$-namely $\{(x_1, ..., x_{n-1}, 0)\}$-has measure zero.

b)Let U be an open set of $\mathbb{R}^m$, and let $f : U \to \mathbb{R}^m$ be a smooth map. Show the if $A \subset U$ is of measure zero, then $f(A)$ is of measure zero.

c)Let U be an open set of $\mathbb{R}^m$, and let $f : U \to \mathbb{R}^n$ be a smooth map. Show if $n > m$, then $f(U)$ has measure zero in $\mathbb{R}^n$.

d)Let M and N be smooth manifolds of dimension m and n respectively. If $f : M \to N$ is a smooth map and $m < n$, show that $f(M)$ has measure zero.

Problem 5. For this problem feel free to quote and use any results from lecture or homework.

Show that if M is a compact k-dimensional manifold, then there exists a map $g : M \to \mathbb{R}^{2k-1}$ that is an immersion except at finitely many points of M.